Review of Signals & Systems, Probability and Noise

Contents

Introduction

In this manual, you have been generally provided materials that have been instructed in Signals & Systems and Probability & Random Process related to ECE489 syllabus. Section 1-4 is particularly accumulated from the references [1] and [2] along with the communication courses taught at NJIT. Section 5, namely Probability, is basically based on the references [4] and [5]. For further information, please see the references.

As a reminder, you have already been familiar to these concepts mentioned in ECE481. The aim of this manual is to introduce the concepts along with the relevance and need for Fundamentals of Communications as well as ECE489. If you feel weak in anything covered here, you must first start studying the basics of ECE321 and ECE333 courses. If you still need a further assistance, please feel free to contact with the instructor.

This manual is not a mandatory task for you; is more supplementary material. However, you will be kept responsible to know the concepts covered here.

1. Classifications of Signals

I. Deterministic vs. Random

A signal can be specified as *deterministic* if it is a specified function of time. As an example,

$$x(t) = \cos(wt + \theta)$$

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

Signals that can take any value with different probabilities with respect to time are defined as *random signals*. These signals can be analyzed for their characteristics such as expected value, variance, probability density function, etc. As an example, the normal distribution is given as

$$\mathrm{X} \sim \mathrm{N}(\mu, \sigma_{\mathrm{x}}^2) = \mathrm{f_x}(x) = \frac{1}{\sqrt{2\pi\sigma_{\mathrm{x}}^2}} e^{\frac{(x-\mu)^2}{2\sigma_{\mathrm{x}}^2}}$$

II. Energy vs. Power

Energy of a signal $x(t)$:

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt \quad (Joules)$$

Any signal with $E_x < +\infty$ i.e. with finite energy, is called *energy signal*. As an example,

$x(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$, $x(t) = sinc(t)$, a speech signal in a finite length, etc.

Power of a signal $x(t)$:

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |x(t)|^2 \, dt \ \ (Watts)$$

A signal $0 < P_x < +\infty$, i.e., finite non-zero power, is a *power signal*. As an example, $x(t) = \cos(wt + \theta)$, periodic pulse train, etc.

Note that:

$$E_x < +\infty \rightarrow \ P_x = 0$$

$$P_x > 0 \rightarrow E_x = +\infty$$

(ex.: Periodic Rectangular Pulse Train)

III. Real vs. Complex

A complex signal has the form

$$(1) \quad z(t) = x(t) + jy(t), \quad \text{Cartesian Representation}$$

$$Re\{z(t)\} = x(t)$$

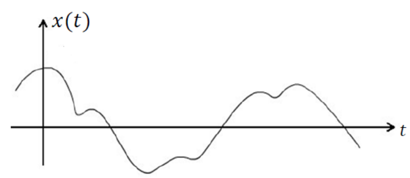
$$Im\{z(t)\} = y(t), \qquad \text{where } x(t) \text{ and } y(t) \text{ are real}$$

$$(2) \quad z(t) = |z(t)|e^{j\theta(t)}, \quad \text{Polar representation}$$

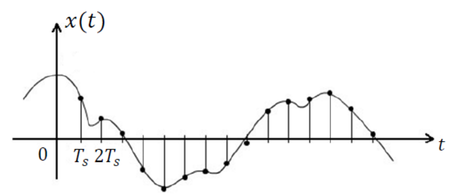
$$|z(t)| \text{ and } \theta(t) \text{ are real}$$

IV. Continuous vs. Discrete

A continuous-time signal:



A discrete-time signal is often obtained by sampling a continuous time signal:
T_s = sampling period



V. Periodic vs. Aperiodic

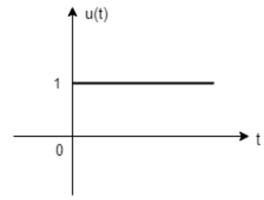
If $x(t)$ is periodic, then

$$x(t) = x(t + nT_0), \quad for\ T_0 \neq 0\ and\ \forall\ integers\ n$$

Some waveforms that you should always remember:

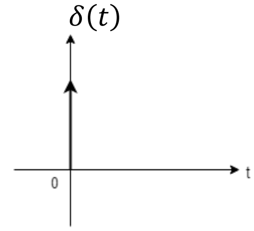
- The Unit Step Function:

  $$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \quad \Longrightarrow$$



- The Unit Impulse Function (delta function):

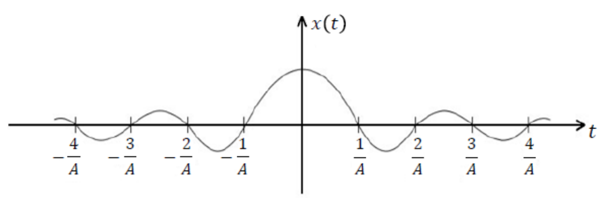
  $$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}, \quad and\ \int_{0^-}^{0^+} \delta(t)dt = 1 \ \Longrightarrow$$



  Recall:

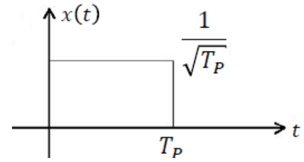
  - $\int_{-\infty}^{+\infty} \phi(t)\delta(t - t_0)dt = \phi(\mathrm{t}_0)$
  - $\int_{0^-}^{0^+} \delta(t)\, dt = 1$
  - $\delta(at) = \frac{1}{|a|}\delta(t)$
  - $\delta(-t) = \delta(t)$
  - $x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$
  - $\delta(t) = u'(t) = \frac{du(t)}{dt}$

- $sinc(At) = \frac{\sin(\pi At)}{\pi At}$ for A >0, $\quad E_x = \frac{1}{A}$

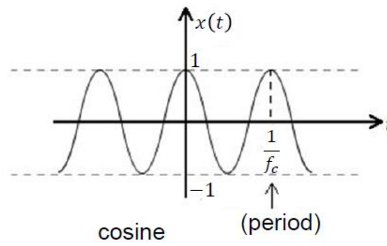


for $A > 1$: "shrinking" $\longrightarrow$ less energy
for $A < 1$: "stretching" $\longrightarrow$ more energy

- A rectangular pulse, $x(t) = \begin{cases} \frac{1}{\sqrt{T_P}}, & t > 0 \\ 0, & t < 0 \end{cases}$, $\quad E_x = 1J$



- $x(t) = \cos(2\pi f_c t), -\infty < t < +\infty, \quad P_x = \frac{1}{2}W$

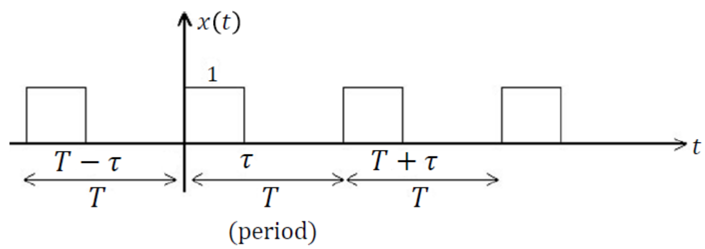


- A periodic pulse train,

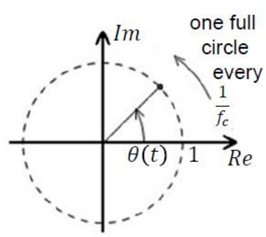
  $P_x = \frac{\tau}{T}$

  For any periodic signal,

  $P_x = \frac{E_x}{T}, \ 0 \le t \le T$

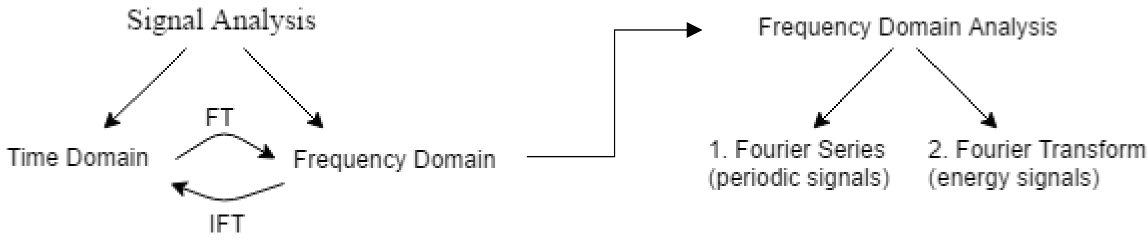


- $z(t) = e^{j2\pi f_c t} = \cos(2\pi f_c t) + j\sin(2\pi f_c t)$
  $Re\{z(t)\} = \cos(2\pi f_c t)$
  $Im\{z(t)\} = \sin(2\pi f_c t)$
  $|z(t)| = 1$
  $\theta(t) = 2\pi f_c t$



2. Signal Analysis

Signals can be analyzed as described below:

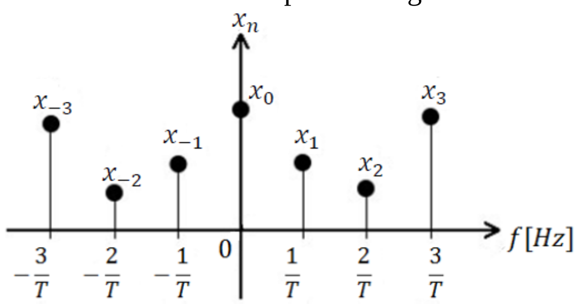


I. Fourier Series

$$x(t) = \sum_{-\infty}^{+\infty} x_n e^{\frac{j2\pi n}{T}t} = \cdots + x_{-1}e^{-\frac{j2\pi 1}{T}t} + x_0 + x_1 e^{\frac{j2\pi 1}{T}t} + \cdots$$

Remark: $e^{\frac{j2\pi n}{T}t}$ "oscillates" at frequency $\frac{n}{T}$

Therefore, we can represent the Fourier Series a periodic signal in the frequency domain as,



Fourier Series Coefficients is calculated as,

$$x_n = \frac{1}{T}\int_0^T x(t)e^{-\frac{j2\pi n}{T}t}dt, \qquad n = \cdots, -1, 0, 1, \ldots$$

Parseval's Theorem

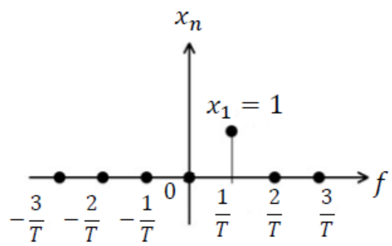
Power can be calculated in time and frequency domains

$$P_x = \frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{-\infty}^{+\infty} |x_n|^2$$

Examples

1) $x(t) = e^{j2\pi f_c(t)} = \cos\big(2\pi f_c(t)\big) + j sin\big(2\pi f_c(t)\big)$, periodic with $T = \frac{1}{f_c}$

   Fourier Series Coefficients will be $x_1 = 1 \ and \ x_n = 0 \ for \ \forall \ n \neq 1$



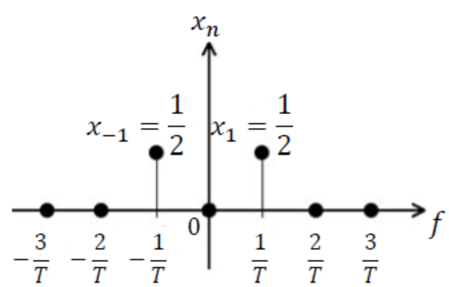
2) $x(t) = \cos(2\pi f_c(t))$, periodic with $T = \frac{1}{f_c}$

   Fourier Series Coefficients will be $x_1 = \frac{1}{2}$ & $x_{-1} = \frac{1}{2} \ and \ x_n = 0 \ for \ \forall \ n \neq \pm 1$

   Equivalently,

   $\cos(2\pi f_c t) = \frac{1}{2} e^{j2\pi f_c t} + \frac{1}{2} e^{-j2\pi f_c t}$

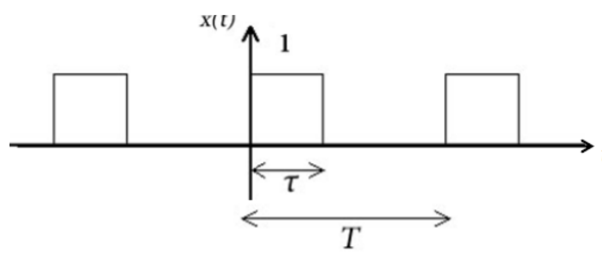
   (Euler's formula)



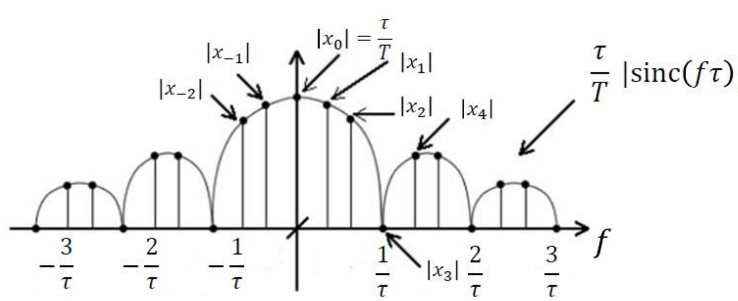
3) Pulse Train

   Periodic with period T

   $x_n = \frac{\tau}{T} sinc(\frac{n}{T}\tau) e^{-\frac{j\pi n}{T} t}$



   Let $\tau = \frac{T}{3}$, then

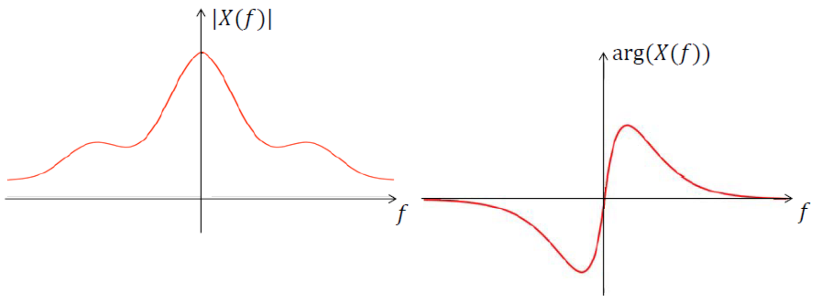


4) Parseval's Theorem

   For $x(t) = \cos(2\pi f_c t)$, then the power is $P_x = \frac{1}{2} = |x_{-1}|^2 + |x_1|^2$

II. Fourier Transform

As it is defined above, *Fourier series* is valid for periodic signals, *Fourier transform* represents an analysis of an energy signal as the continuous spectrum of frequencies.



Fourier transform:

$$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi f t} dt$$

Inverse Fourier Transform:

$$x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{+\infty} X(f)\, e^{j2\pi f t} df$$

Properties of the Fourier transform

1) If $x(t)$ is real, then

$X(f) = X^*(-f)$, similarly

$$\left.\begin{aligned} |X(f)| &= |X(-f)| \\ arg\big(X(f)\big) &= -\arg(X(-f)) \end{aligned}\right\} \therefore \textit{Hermitian Symmetry}$$

Ex.: Rectangular signal, $|x(t)|$ sinc, x^2, cosine, etc.

2) If $x(t)$ is real and even (i.e. $x(t) = x(-t)$), then

$X(f)$ is real and even ($X(f) = X(-f)$

Ex.: Sinc, rectangular signal centered at t=0

3) If $x(t)$ is real and odd (i.e. $x(t) = -x(-t)$), then

$$X(f) \text{ is real and odd (i.e. } X(f) = -X(-f))$$

Ex.: $sine, \text{erf}(x)$, etc.

4) Rayleigh Theorem,

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |X(f)|^2 df,$$

It is clearly seen that energy can be calculated both in time and frequency domains.

5) Delay,

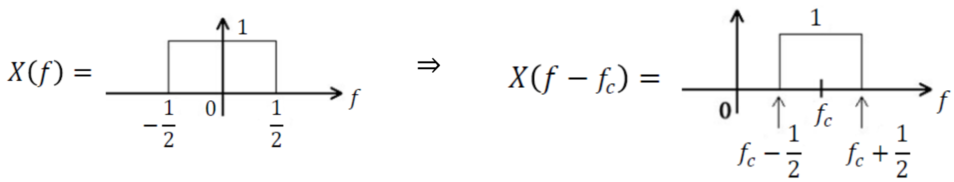
$$\mathcal{F}\{x(t-\tau)\} = X(f)e^{-j2\pi f\tau}$$

Please note that a delay in time domain results in a linear phase shift in the frequency domain.

6) Frequency translation,

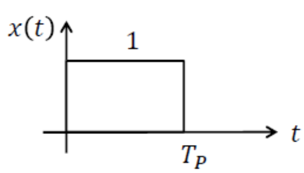
$$\mathcal{F}\left\{x(t)e^{-j2\pi f_c t}\right\} = X(f - f_c)$$

Please note that multiplying a function by $e^{j2\pi f_c t}$ is called "upconversion" in communication systems.

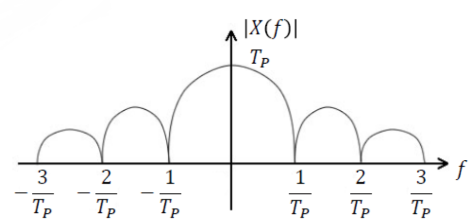


Examples

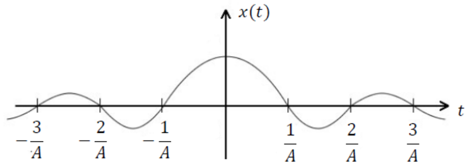
1) $$x(t) = \begin{cases} 1, & 0 \le t \le T_p \\ 0, & elswhere \end{cases}$$



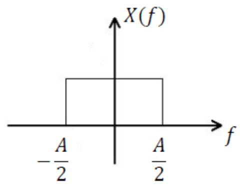
$$X(f) = \int_0^{T_p} e^{-2\pi f t}\, dt = T_p sinc(fT_p)e^{-j\pi t T_p}$$



2) $x(t) = Asinc(At)$



$$X(f) = \begin{cases} 1, |f| \le \frac{A}{2} \\ 0, \quad elsewhere \end{cases}$$



$$E_x = A$$

3. Correlation and Energy Spectrum

Correlation

Correlation function of a signal $x(t)$,

$$R_x(\tau) = \int_{-\infty}^{+\infty} x(t)x^*(t-\tau)dt$$

It measures the correlation between $x(t)$ and $x(t-\tau)$

Properties of correlation function

1) $R_x(o) = \int_{-\infty}^{+\infty} |x(t)|^2 dt = E_x$

2) $R_x(\tau) = R_x^*(-\tau)$ $\quad\quad \therefore Hermitian\ Symmetry$
3) $|R_x(\tau)| \le R_x(0)\ for\ \tau \ne 0$

Energy Spectrum

Energy spectrum of a signal $x(t)$

$$G_x(f) = |X(t)|^2 \ \ (energy\ spectral\ density) \quad [\frac{J}{Hz}]$$

It measures the energy of the signal at frequency f.

$$E_x = \int_{-\infty}^{+\infty} G_x(f)df = \int_{-\infty}^{+\infty} |X(f)|^2 df$$

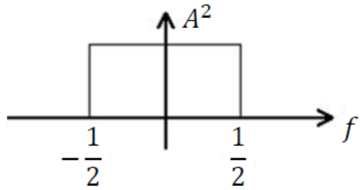
Please note that the relationship between correlation and the energy spectrum density:

$$\mathcal{F}\{R_x(\tau)\} = G_x(f)$$

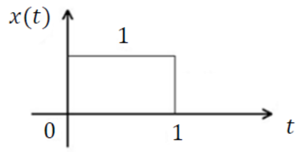
Examples

1) $x(t) = Asinc(t) \Longrightarrow G_x(f) =$

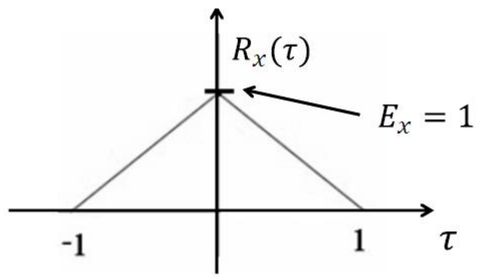
$E_x = A^2$



2) $x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & elsewhere \end{cases}$



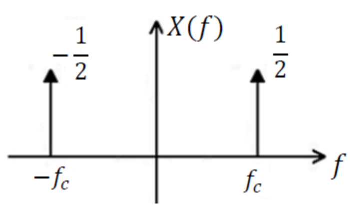
$$R_x(\tau) = \int_{-\infty}^{+\infty} x(t)x(t-\tau)dt = \begin{cases} 1-|\tau|, & for\ |\tau| \le 1 \\ 0, & otherwise \end{cases}$$



3) Fourier series as a Fourier transform

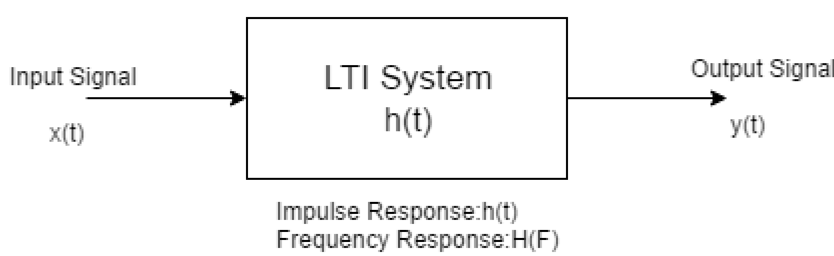
$X(f) = \sum_{-\infty}^{+\infty} x_n \delta\left(f - \frac{n}{T}\right)$

Let $x(t) = \cos(2\pi f_c t)$, then



4. LTI Systems

Physical systems are characterized by the input/output relations.



Linear time-invariant (LTI) systems both satisfy the linearity and the time-invariance conditions [2].

Linearity

A linear system is one of in which superposition holds, i.e.,

$$ax_1(t) + bx_2(t) \to ay_1 + by_2$$

Time-Invariance

A time-invariant system is one in which a time shift in the input only changes the output by time a shift, i.e.,

$$x(t-\tau) \to y(t-\tau)$$

- In time domain: The output of the linear system is the convolution of the input signal with the impulse response, i.e.,

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\lambda) d\lambda$$

- Frequency Domain: The Fourier transform (or Laplace) of the impulse response is donated the frequency response (transfer function), $H(f) = \mathcal{F}\{h(t)\}$ and by the convolution theorem for the energy signals, we have

$$Y(f) = H(f)X(f)$$

An LTI system's energy spectrum has the form

$$G_y(f) = H(f)H^*(f)G_x(f) = |H(f)|^2 G_x(f) = G_h(f)G_x(f)$$

Examples

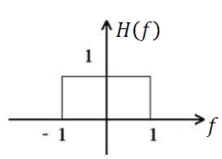
1) $h(t) = \delta(t) \to H(f) = 1$, then

$Y(f) = H(f)X(f) \to Y(f) = X(f)$, similarly $y(t) = x(t)$

2)
a. An LTI system has the impulse response, is defined as

$h(t) = 2sinc(2t)$, and the input $x(t) = 4sinc(4t)$. Find the output of the system.

- $H(f) = \begin{cases} 1, & -1 \le t \le 1 \\ 0, & otherwise \end{cases}$



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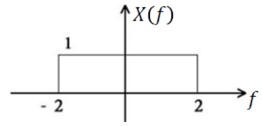
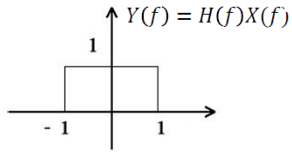
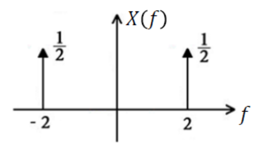
$$X(f) = \begin{cases} 1, & -2 \le t \le 2 \\ 0, & otherwise \end{cases}$$



The output,  $\rightarrow y(t) = 2sinc(2t)$

b. Same filter as defined above, but the input $x(t) = \cos(4\pi t)$, find the output $y(t)$.

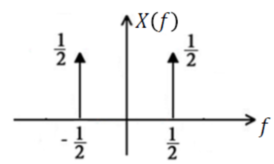
$$X(f) = \begin{cases} -\frac{1}{2}, \frac{1}{2}, & for\ f = -2,2\ respectively \\ 0, & otherwise \end{cases}$$



Therefore, $Y(f) = H(f)X(f) = 0 \rightarrow y(t) = 0$

c. Now the input is $x(t) = \cos(\pi t)$

$$X(f) = \begin{cases} -\frac{1}{2}, \frac{1}{2}, & for\ f = -\frac{1}{2}, \frac{1}{2}\ respectively \\ 0, & otherwise \end{cases}$$



Therefore, $Y(f) = X(f) \rightarrow y(t) = x(t) = \cos(\pi t)$

Finding Sampling Frequency of the Signals

Nyquist-Shannon Theorem: $f_s = \frac{1}{T_s} \ge 2\times$highest frequency of $X(f)$

In Matlab/Simulink, it is suggested that $f_s = \frac{1}{T_s} \ge 10\times(2\times$highest frequency of $X(f))$

5. Probability and Noise in Communication Systems

Why Probability?

Probability is the mathematical tool for communications theory. Consider a radio communication system where the received signal is a random process in nature; message and interference are random as well as delay, phase, fading, etc. [3] Thus, the probability concept is crucial for communications engineering.

I. Probability Concept

To refresh our memory, let's start with an example.

Consider a fair dice rolled once, the sample space for possible outcomes will be

$$\Omega = \{1,2,3,4,5,6\}$$

Assume that "A" is scenario that the outcome is greater and equal to 4, then the sample space of the event A is

$$A = \{4,5,6\}$$

- The probability of A,

$$P(A) = \frac{total\ \#\ of\ outcomes\ of\ A}{tatal\ \#\ of\ \Omega} = \frac{3}{6} = 0.5$$

Let an event B is the even numbers selected, then

$$B = \{2,4,6\}$$

The probability of B,

$$P(B) = \frac{3}{6} = 0.5$$

- The conditional probability is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Therefore,

$$P(A|B) = \frac{\{4,6\}}{\{2,4,6\}} = \frac{2}{3} \approx 0.667$$

- The complement of the event B is the odd outcomes,

$$A^c = \{1,3,5\}$$

- The total probability law is expressed as

$$P(A) = P(B)P(A|B) + P(B^c)P(A|B^c)$$

- Three conditions must be satisfied to understand the random events that characterize communication systems performance [4]:
  - For any event A, $P(A) \geq 0$
  - $P(\Omega) = 1$
  - If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$

- Bayes Rule is expressed as

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

When a signal transmitted through a communication channel, there are two types of imperfections that cause the received signal to be different from the transmitted signal, namely;

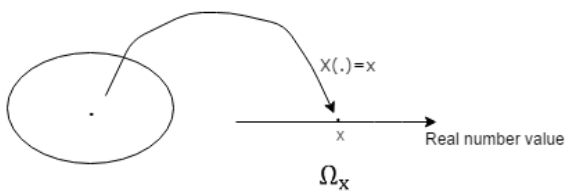
i. Deterministic in nature, such as linear and nonlinear distortions, inter-symbol-interference (ISI), etc.
ii. Nondeterministic, such as addition of noise, interference, multipath fading, etc.

For these nondeterministic phenomena, a random model is required [5].

II. Random Variables

A random variable is a mapping function whose domain is a sample space and whose range is some set of real numbers [6]:

$$X = X(\Omega)$$



Probability of specific event is expressed as

$$P(\text{"."} \in \Omega{:}\, \mathrm{X}(.) < \mathrm{x}) \rightarrow P(X < \mathrm{x})$$

Random variables are studied in two forms, namely, discrete and continuous. However, you have already seen their properties in other courses. Therefore, we will just introduce the probability mass functions (PMF) (discrete) and density functions (PDF) (continuous) of some distributions.

Mean and Variance

Mean (or Expected Value) is also called DC level is expressed as:

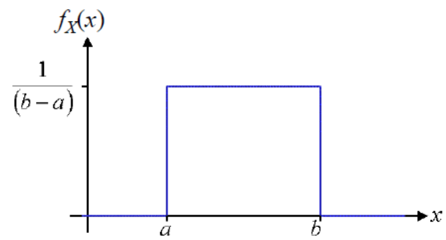
$$E[X] = \mu_X = \int_{-\infty}^{+\infty} x f_x(x) dx$$

Variance (power for zero mean signals):

$$\sigma_X = E[X - \mu_X] = \int_{-\infty}^{+\infty} (x - \mu_X)^2 f_X(x) dx = E[X^2] - \mu_X^2$$

Uniform

PMF: $$p_x(k) = \begin{cases} \frac{1}{n}, & k = 1, \dots, n \\ 0, & otherwise \end{cases}$$



PDF: $$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & otherwise \end{cases}$$

$$E_X = \frac{a+b}{2}$$

$$\sigma_X^2 = \frac{(b-a)^2}{2}$$

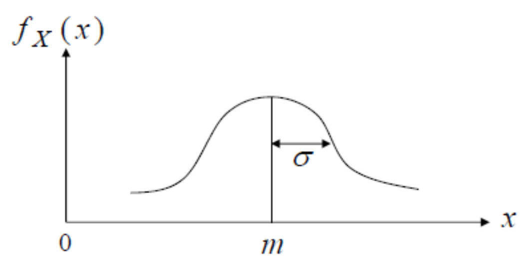
Gaussian

PDF: $$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \text{ for } -\infty < x < +\infty$$

$m = E[X]$

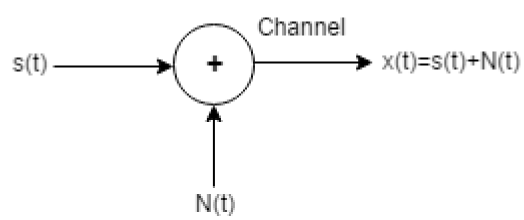
$\sigma_X^2 = \sigma^2$

$\sigma : standard\ deviation$



III. Noise in Communication Systems

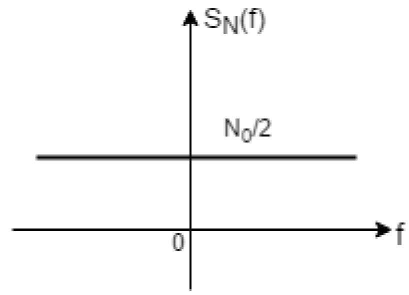
Noise is the undesired waves that disturb the transmission of signals. Basically, the sources of noise are: external (atmospheric, interference, etc.) and internal (generated by the communication devices, i.e. electron's randomly fluctuations, etc.).



White Noise

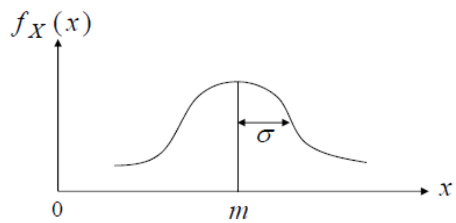
White Noise (WN) has zero mean, stationary, and occupies all frequencies. The power spectrum density of WN is:

$$S_N(f) = \frac{N_0}{2}, \quad -\infty < f < +\infty$$



Gaussian Noise

Gaussian Noise (GN) is the distribution at any instant is Gaussian. GN can be colored.



Typically, the noise is Additive White Gaussian (AWGN) in communication systems.

Note that the WN is constant over an infinite bandwidth that does not mean Gaussian waveform. In addition, the GN can be colored.

During the ECE489, you will quite often use AWGN channel in the simulations.

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